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Note

Resolvable packings $\tilde{R}$MP$(3, 2; n, n-3)$ and coverings $\tilde{R}$MC$(3, 2; n, n-2)$ ☆

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Abstract

Let $n \equiv k-1, 0$ or $1 \pmod k$. An $\tilde{R}$MP$(k, \lambda; n, m)$ (resp. $\tilde{R}$MC$(k, \lambda; n, m)$) is a resolvable packing (resp. covering) with maximum (resp. minimum) possible number m of parallel classes which are mutually distinct, each parallel class consists of $\lfloor (n-k+1)/k \rfloor$ blocks of size k and one block of size $n - k\lfloor (n-k+1)/k \rfloor$, and its leave (resp. excess) is a simple graph. Such designs can be used to construct certain uniform designs which have been widely applied in industry, system engineering, pharmaceutics, and natural sciences. In this paper, direct and recursive constructions are discussed for such designs. The existence of an $\tilde{R}$MP$(3, 2; n, n-3)$ and an $\tilde{R}$MC$(3, 2; n, n-2)$ for $n \equiv 1 \pmod 3$ is established with $n \geqslant 16$.

Keywords: Uniform design; Resolvable packing/covering; Frame

1. Introduction

Let n and λ be positive integers. A *packing* (resp. *covering*) of X is a collection of subsets of X (called *blocks*) such that any pair of distinct points from X occur together in at most (resp. at least) λ blocks in the collection. Denote by P$(K, \lambda; n)$ (resp. C$(K, \lambda; n)$) a packing (resp. covering) on n points with block sizes from K.

For any pair $e = \{x, y\}$ of distinct points , let $m(e)$ be the number of blocks containing e. The *leave* (resp. *excess*) of a packing (resp. covering) P$(K, \lambda; n)$ (resp. C$(K, \lambda; n)$) is the multigraph spanned by all pairs e of distinct points with multiplicity $\lambda - m(e)$ (resp. $m(e) - \lambda$).

A packing or a covering is called *resolvable* if its block set admits a partition into *parallel classes*, each parallel class being a partition of the point set X. Denote by RP$(K, \lambda; n, m)$ (resp. RC$(K, \lambda; n, m)$) a resolvable P$(K, \lambda; n)$ (resp. C$(K, \lambda; n)$) with m parallel classes. We write RP$(k, \lambda; n, m)$ (resp. RC$(k, \lambda; n, m)$) instead of RP$(K, \lambda; n, m)$ (resp. RC$(K, \lambda; n, m)$) when $K = \{k\}$.

Let $n \equiv k-1, 0$ or $1 \pmod k$. An $\tilde{R}$MP$(k, \lambda; n, m)$ (resp. $\tilde{R}$MC$(k, \lambda; n, m)$) is a resolvable packing (resp. covering) RP$(\{k-1, k, k+1\}, \lambda; n, m)$ (resp. RC$(\{k-1, k, k+1\}, \lambda; n, m)$) which satisfies the following properties:

1. it contains the maximum (resp. minimum) possible number m of parallel classes which are mutually distinct;
2. each parallel class consists of $\lfloor (n-k+1)/k \rfloor$ blocks of size k and one block of size $n - k\lfloor (n-k+1)/k \rfloor$;
3. its leave (resp. excess) is a simple graph, that is, $\lambda - m(e) \leqslant 1$ (resp. $m(e) - \lambda \leqslant 1$) for any pair e of distinct points.

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Both $\tilde{R}\mathrm{MP}(k, \lambda; n, m)$s and $\tilde{R}\mathrm{MC}(k, \lambda; n, m)$s were first introduced by Fang et al. [7]. They can be used in the construction of uniform designs in statistics, which have been widely applied in industry, system engineering, pharmaceutics, and natural sciences [5,6,19].

Theorem 1.1 (*Fang et al. [7], Theorem 3.1*). *Let n, k, λ and m are positive integers, and $n \equiv r \pmod k$ where $r \in \{0, 1, k-1\}$. If there exists an $\tilde{R}\mathrm{MP}(k, \lambda; n, m)$ (or $\tilde{R}\mathrm{MC}(k, \lambda; n, m)$), then there exists a uniform design $U_n(q^m)$, where $q = \lfloor (n-k+1)/k \rfloor + 1$.*

In the particular case where the $\tilde{R}$MP and $\tilde{R}$MC are exact, that is, the leave and excess are empty, we simply write $\tilde{R}\mathrm{B}(k, \lambda; n, m)$ for both $\tilde{R}\mathrm{MP}(k, \lambda; n, m)$ and $\tilde{R}\mathrm{MC}(k, \lambda; n, m)$. In the literature, an $\tilde{R}\mathrm{MP}(k, 1; n, m)$ is also called a Kirkman packing design or a Kirkman school project design when $k = 3$, an $\tilde{R}\mathrm{MC}(k, 1; n, m)$ is called a Kirkman covering design. When $k = 3$ and $\lambda \in \{1, 2\}$, the existence of an $\tilde{R}\mathrm{MP}(3, 1; n, m)$ or $\tilde{R}\mathrm{MC}(3, 1; n, m)$ has been solved for every positive integer n with few possible exceptions [15,16,12,2,4,14,1]; the existence of an $\tilde{R}\mathrm{MP}(3, 2; n, m)$ or $\tilde{R}\mathrm{MC}(3, 2; n, m)$, $n \equiv 0, 2, 3, 4, 5 \pmod 6$, has been solved with some possible exceptions [17,7]. There are also some results on $\tilde{R}\mathrm{MP}(4, \lambda; n, m)$ and $\tilde{R}\mathrm{MC}(4, \lambda; n, m)$ for $\lambda \in \{1, 2\}$ [11,13,9,10,7]. The main known results concerning $\tilde{R}\mathrm{MP}(3, \lambda; n, m)$ and $\tilde{R}\mathrm{MC}(3, \lambda; n, m)$ are as follows.

Theorem 1.2. *There exist*

1. *an $\tilde{R}\mathrm{MP}(3, 1; n, \lfloor n - 1/2 \rfloor)$ and an $\tilde{R}\mathrm{MC}(3, 1; n, n/2)$ when $n \equiv 0 \pmod 3$ and $n \notin \{6, 12\}$;*
2. *an $\tilde{R}\mathrm{MP}(3, 1; n, \lfloor n - 3/2 \rfloor)$ when $n \equiv 1 \pmod 3$ and except for $n \in \{1, 4, 7, 10, 13\}$ and possibly except for $n = 19$;*
3. *an $\tilde{R}\mathrm{MC}(3, 1; n, \lfloor n - 1/2 \rfloor)$ when $n \equiv 1 \pmod 3$ and $n \notin \{16, 67\}$;*
4. *an $\tilde{R}\mathrm{MP}(3, 1; n, \lfloor n/2 \rfloor)$ when $n \equiv 2 \pmod 3$ and $n \notin \{5, 11\}$;*
5. *an $\tilde{R}\mathrm{MC}(3, 1; n, \lfloor n + 1/2 \rfloor)$ when $n \equiv 2 \pmod 3$ and $n \notin \{5, 11\}$;*
6. *an $\tilde{R}\mathrm{B}(3, 2; n, n)$ when $n \equiv 0 \pmod 3$ and $n \geqslant 9$;*
7. *an $\tilde{R}\mathrm{B}(3, 2; n, n)$ when $n \equiv 2 \pmod 3$ and $n \geqslant 8$;*
8. *an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ when $n \equiv 4 \pmod 6$, $n \geqslant 16$ and $n \notin \{28, 34, 40, 46, 58, 70, 82, 94, 142\}$.*

In this paper, we mainly deal with the existence of an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for $n \equiv 1 \pmod 6$, we also solve some left possible exceptions in other infinity classes. Direct and recursive constructions are discussed for an $\tilde{R}$MP and an $\tilde{R}$MC. The existence of an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for $n \equiv 1 \pmod 3$ is established with $n \geqslant 16$.

Theorem 1.3. *There exist an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for every $n \equiv 1 \pmod 3$ and $n \geqslant 16$.*

2. Direct construction using *DF(n, K, 2)

In this section, we construct some small $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$s some of which will be used as input designs in recursive constructions of the next section. These $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$s can be obtained from a special difference family *DF$(n-4, K, 2)$.

Definition 2.1. Suppose $n \equiv 0 \pmod 3$. Let G be the additive group of Z_n. Let $S = (B_1, B_2, \ldots, B_s)$ be a partition of G with $|B_i| \in K$, $1 \leqslant i \leqslant s$. B_i are called base blocks. Then S is called a special difference family *DF$(n, K, 2)$ if each non-zero element $g \neq n/3$ of G occurs equally twice in the sum $\triangle S$ of the difference lists

$$\triangle B_j := (b - b' : \ b, b' \in B_j, \ b \neq b')$$

of the base blocks B_j, and $g = n/3$ occurs exactly once in $\triangle S$.

The *type* of a *DF$(n, K, 2)$ is the multiset $\{|B_j| : B_j \in S\}$. We denote the type by $2^{u_2}3^{u_3}, \ldots,$ where there are precisely u_i occurrences of i, $i \geqslant 2$.

Lemma 2.1. *Let $n \equiv 0 \pmod 3$, $n \geqslant 12$ and $u = (n-12)/3$. If there exists a* *DF$(n, K, 2)$ *with type* $2^4 3^u 4^1$, *then there exist an* $\tilde{R}$MP$(3, 2; n+4, n+1)$ *and an* $\tilde{R}$MC$(3, 2; n+4, n+2)$.

Proof. Take the point set $V = Z_n \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. Suppose $S = (B_1, B_2, \ldots, B_s)$ is a *DF$(n, K, 2)$ on Z_n with type $2^4 3^u 4^1$. Let $|B_i| = 2$ for $1 \leqslant i \leqslant 4$. Add ∞_i to the base block B_i, we may obtain a new block B_i'. It is obvious that $(B_1', B_2', B_3', B_4', B_5, \ldots, B_s)$ is a partition of V, that is, they form a parallel class P_1. Thus n parallel classes can be generated from P_1 by $+1 \bmod n$, where $\infty_i + x = \infty_i$. These n parallel classes form an RP$(\{3, 4\}, 2; n+4, n)$, the leave contains $(n/3)K_3$ and two K_4. By the definition of a *DF$(n, K, 2)$, these $(n/3)K_3$ is a partition of Z_n. So, we can obtain a new parallel class P_{n+1} on V which contains $(n/3)K_3$ and one K_4. It is easy to check that the $n+1$ parallel classes form an $\tilde{R}$MP$(3, 2; n+4, n+1)$ on V whose leave is a K_4 on $\{\infty_1, \infty_2, \infty_3, \infty_4\}$. Furthermore, an $\tilde{R}$MC$(3, 2; n+4, n+2)$ can be obtained by adding one more parallel class P_{n+2} to the $\tilde{R}$MP constructed above. The blocks of P_{n+2} are as follows: $\{3i, 3i+1, 3i+2\} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, $i \in Z_{n/3}$. $\square$

Remark. Note that no pair of infinity points appears in the parallel classes $P_1, \ldots, P_n$. This property is important for the later recursive constructions.

Lemma 2.2. *There exists a* *DF$(n, K, 2)$ *with type* $2^4 3^u 4^1$ *for each* $n \in \{12, 24, 30, 36, 42, 54, 66, 78\}$, *where* $u = (n-12)/3$.

Proof. The base blocks for each n are listed as follows:

$n=12$	1 2 3 6	4 10	5 8	0 7	9 11				
$n=24$	1 2 3 5	4 7 13	6 10 19	8 15 20	9 17 22	11 21	12 18	0 14	16 23
$n=30$	1 2 3 5	4 7 11	6 12 21	8 13 24	9 18 26	10 22 27	14 20 28		
	15 25	16 23	17 29	0 19					
$n=36$	1 2 3 5	4 7 11	6 12 25	8 13 23	9 14 28	10 16 30	0 15 27		
	17 24 35	18 26 34	19 29	20 33	21 32	22 31			
$n=42$	1 2 3 5	4 7 11	6 12 22	8 13 28	9 14 30	10 16 33	15 23 38		
	17 24 35	0 18 32	19 27 36	20 31 40	21 34	25 37	26 39	29 41	
$n=54$	1 2 3 5	21 33 47	0 25 35	8 13 19	9 16 24	10 18 28	14 23 48		
	4 7 38	6 11 39	12 26 42	15 22 45	17 29 44	20 37 41	27 36 49		
	30 46 52	31 50	32 43	34 51	40 53				
$n=66$	1 2 3 5	30 46 62	31 44 64	34 41 59	28 39 58	10 18 27	14 23 33		
	15 25 37	20 32 60	4 7 36	6 11 42	8 12 54	9 16 57	13 19 40		
	17 22 43	21 29 52	24 47 53	26 50 65	35 49 63	38 55	45 56		
	48 61	0 51							
$n=78$	1 2 3 5	31 49 77	38 58 65	39 43 62	41 47 72	24 44 50	19 27 67		
	0 17 40	20 32 45	21 34 48	22 36 51	26 42 71	4 7 28	6 11 73		
	8 13 74	9 16 53	10 18 57	12 23 60	14 29 64	15 25 61	30 66 75		
	33 55 76	35 59 69	37 56	46 68	52 70	54 63			

$\square$

Lemma 2.3. *There exists a* *DF$(n, K, 2)$ *with type* $2^4 3^u 4^1$ *for each* $n \in \{15, 21, 27, 33, 39, 45, 51, 57, 69, 81\}$, *where* $u = (n-12)/3$.

Proof. The base blocks for each n are listed as follows:

$n=15$	1 2 3 6	4 7 11	5 13	8 14	0 9	10 12		
$n=21$	1 2 3 5	4 7 13	6 11 16	8 15 19	9 17	10 18	0 12	14 20
$n=27$	1 2 3 5	4 7 11	6 12 20	8 18 23	9 17 22	10 19 26	13 25	
	14 24	15 21	0 16					
$n=33$	1 2 3 5	4 7 11	6 12 21	8 13 26	9 14 25	10 22 29	15 23 32	
	16 24 30	17 27	18 28	19 31	0 20			

$n=39$	1 2 3 5	4 7 11	6 12 22	8 13 26	9 14 29	10 16 31	15 23 37
	0 17 27	18 30 38	19 28 35	20 34	21 32	24 33	25 36
$n=45$	1 2 3 5	4 7 11	6 12 22	8 13 29	9 14 32	10 16 37	15 23 38
	17 24 36	18 26 35	19 33 44	0 20 34	21 31 40	25 42	27 39
	28 41	30 43					
$n=51$	1 2 3 5	22 34 46	0 23 32	15 25 45	9 16 24	10 18 36	4 7 20
	6 11 21	8 12 38	13 19 41	14 27 43	17 35 49	26 33 50	28 39 48
	29 40	30 44	31 37	42 47			
$n=57$	1 2 3 5	22 34 52	29 41 54	20 30 55	9 16 24	10 18 27	14 23 38
	15 25 44	4 7 31	6 11 47	12 26 46	8 13 39	17 21 37	19 32 53
	28 45 56	33 40 51	0 35	36 50	42 48	43 49	
$n=69$	1 2 3 5	31 47 65	39 58 62	36 43 68	17 28 52	22 30 48	14 23 33
	15 25 37	20 32 45	21 34 61	4 7 24	6 11 42	8 13 44	9 16 56
	10 18 63	12 26 54	19 40 51	27 57 66	29 35 55	38 53 59	0 41
	46 60	49 64	50 67				
$n=81$	1 2 3 5	38 58 79	39 57 80	47 66 77	40 64 71	24 29 54	12 18 41
	30 49 73	20 32 45	21 34 48	22 36 51	26 42 60	28 44 65	4 7 43
	6 10 16	8 13 56	9 17 62	11 19 50	14 23 69	15 25 74	27 53 75
	31 63 78	33 61 68	35 52 72	37 46	55 67	59 76	0 70

□

Combining Lemma 2.1 with Lemmas 2.2 and 2.3, we get the following two results.

Lemma 2.4. *There exist an* $\tilde{R}$MP$(3, 2; n, n-3)$ *and an* $\tilde{R}$MC$(3, 2; n, n-2)$ *for each* $n \in \{16, 28, 34, 40, 46, 58, 70, 82\}$.

Lemma 2.5. *There exist an* $\tilde{R}$MP$(3, 2; n, n-3)$ *and an* $\tilde{R}$MC$(3, 2; n, n-2)$ *for each* $n \equiv 1 \pmod 6$, $19 \leqslant n \leqslant 61$ *and* $n \in \{73, 85\}$.

3. Main results

A group-divisible design of index λ, denoted by (K, λ)-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties: (i) X is a finite set of *points*, (ii) $\mathcal{G}$ is a partition of X into subsets called *groups*, (iii) $\mathcal{B}$ is a set of subsets of X with sizes from K (called *blocks*), such that a group and a block contain at most one common point, and every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of a GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We denote the type by $1^{u_1}2^{u_2}, \ldots,$ where there are precisely u_i occurrences of i, $i \geqslant 1$. A (k, λ)-GDD is a GDD in which all the blocks have size k. *A transversal design* TD(k, λ, n) is a (k, λ)-GDD of type n^k. It is *idempotent* if it contains a parallel class of blocks.

A GDD$(X, \mathcal{G}, \mathcal{B})$ is called *frame resolvable* if its block set $\mathcal{B}$ can be partitioned into *frame* parallel classes, each class being a partition of $X - G_j$ for some $G_j \in \mathcal{G}$. A (k, λ)-frame is a frame resolvable (k, λ)-GDD in which all the blocks have size k. It is well known that to each G_j there are exactly $\lambda|G_j|/(k-1)$ frame parallel classes of triples that partition $X - G_j$. The groups in a (k, λ)-frame are often considered as holes. The existence of $(3, \lambda)$-frame has been solved completely.

Theorem 3.1 (*Stinson [18]*). *There exists a* $(3, \lambda)$*-frame of type* g^u *if and only if* $u \geqslant 4$, $\lambda g \equiv 0 \pmod 2$ *and* $g(u-1) \equiv 0 \pmod 3$.

For later use, we need the following weighting construction for frames.

Lemma 3.2 (*Furino et al. [8]*). *Suppose that there is a* $(K, 1)$-GDD *of type* $g_1^{t_1} g_2^{t_2} \ldots g_m^{t_m}$ *and that for each* $k \in K$ *there is a* (w, λ)*-frame of type* h^k. *Then there is a* (w, λ)*-frame of type* $(hg_1)^{t_1}(hg_2)^{t_2} \ldots (hg_m)^{t_m}$.

Lemma 3.3. *For each v, $v \equiv 0 \pmod 3$, $v \geqslant 144$, there is a $(3,2)$-frame of type $12^a 15^b$, where $v = 12a + 15b$, $b \geqslant 4$, $a \geqslant 0$.*

Proof. Let $(n_0, n_1, \ldots)$ be the infinite sequence of integers defined as follows: $n_0 = 11$, $n_j = 11 + j$ for $j \geqslant 1$. From [3, p.126], an idempotent TD $(5, 1, n_i)$ exists for each n_i, $i \geqslant 0$. Let $t = v/3$. Since $v \geqslant 144$, we have $t \geqslant 48$. There exists an integer n_i from the sequence so that $4n_i + 4 \leqslant t \leqslant 5n_i$. This can always be done because $5n_i \geqslant 4n_{i+1} + 4$ for all $i \geqslant 0$. Let $t = 5a + 4b$, where $a = n_i - b$, $0 \leqslant b \leqslant n_i - 4$. Form an idempotent TD $(5, 1, n_i)$ with groups $G_1, \ldots, G_5$ and blocks $B_1, \ldots, B_{n_i}$ in one parallel class. Delete $n_i - a$ points in G_5 that lie in $B_{a+1}, \ldots, B_{n_i}$. Taking the truncated blocks $B_1, \ldots, B_{n_i}$ as groups, we have formed a GDD of type $5^a 4^b$, having all blocks of size at least four. We may apply Lemma 3.2 with weight $h = 3$ to obtain a $(3,2)$-frame of type $12^a 15^b$, where the input $(3,2)$-frames of type 3^4 and 3^5 come from Theorem 3.1. □

Lemma 3.4. *Suppose that there exist a $(3,2)$-frame of type $h_1^{t_1} h_2^{t_2} \ldots h_m^{t_m}$ and a $^*\mathrm{DF}(h_i, K, 2)$ with type $2^4 3^u 4^1$ for each $1 \leqslant i \leqslant m$. Then there exist an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$, where $n = 4 + \sum_{i=1}^{m} h_i$.*

Proof. For each $1 \leqslant i \leqslant m$, there are h_i frame parallel classes missing the group of size g_i, denote these parallel classes by $Q_{i,1}, \ldots, Q_{i,h_i}$. Similarly to the proof of Lemma 2.1, we can add four points $\infty_1, \ldots, \infty_4$ to the group and construct an $\tilde{R}\mathrm{MP}(3, 2; h_i + 4, h_i + 1)$ whose parallel classes are $P_{i,1}, \ldots, P_{i,h_i}, P_{i,h_i+1}$, each of which contains one block of size four. Let $F_{i,j} = P_{i,j} \cup Q_{i,j}$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant h_i$. Let $F' = \bigcup_{i=1}^{m} P_{i,h_i+1}$. Thus we have obtained $1 + \sum_{i=1}^{m} h_i = n - 3$ parallel classes on the whole point set, it is easy to check that they form an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$.

Next by Lemma 2.1, we can construct a parallel class P_{i,h_i+2} for each group of size h_i. Let $F'' = \bigcup_{i=1}^{m} P_{i,h_i+2}$. Add this new parallel class to the $n-3$ parallel classes of the above $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$, we get an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$. □

Combine Lemmas 3.3, 3.4, 2.2 and 2.3, we have the following lemma.

Lemma 3.5. *There exist an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for each $n \equiv 1 \pmod 3$, $n \geqslant 148$.*

By Lemma 3.5, it remains to consider the orders $16 \leqslant n \leqslant 145$.

Lemma 3.6. *There exist an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for each $n \equiv 1 \pmod 3$, $88 \leqslant n \leqslant 145$ and $n \in \{67, 79\}$.*

Proof. By Lemmas 3.2 and 3.4, we only need to construct some GDDs on $(n-4)/3$ points having all blocks and groups of size at least four, where the required input designs $^*\mathrm{DF}(m, K, 2)$ come from Lemmas 2.2 and 2.3. These GDDs can be obtained from an idempotent TD(7, 1, 7) or an idempotent TD(5, 1, 5). We show them in the following table.

n	$(n-4)/3$	$(K,1)$-GDD of type	$0 \leqslant a, b, c, d$
[124, 151]	[40,49]	$7^4 a^1 b^1 c^1$	$4 \leqslant a, b, c \leqslant 7$
[112, 121]	[36,39]	$6^4 5^a 4^b$	$a + b = 3$
[100, 119]	[32,35]	$7^4 a^1$	$4 \leqslant a \leqslant 7$
[88, 97]	[28,31]	$4^6 a^1$	$4 \leqslant a \leqslant 7$
79	25	5^5	
67	21	$4^4 5^1$	

□

Combining Theorem 1.2(8), Lemmas 2.4, 2.5 and Lemmas 3.5, 3.6, we obtain our main results.

Theorem 3.7. *There exist an $\tilde{R}\mathrm{MP}(3, 2; n, n-3)$ and an $\tilde{R}\mathrm{MC}(3, 2; n, n-2)$ for every $n \equiv 1 \pmod 3$ and $n \geqslant 16$.*

4. Concluding remarks

It is not difficult to see that there does not exist an $\tilde{R}$MP$(3, 2; n, n-3)$ and an $\tilde{R}$MC$(3, 2; n, n-2)$ for $n \in \{1, 4, 7\}$.

Lemma 4.1. *There exists an* $\tilde{R}$MP$(3, 2; 10, 7)$.

Proof. Take the point set $V = \{1, 2, \ldots, 10\}$. We list the required seven parallel classes as follows:

1 2 3 4	1 2 5 8	1 3 6 9	1 4 7 10	2 3 7 10	2 4 6 9	3 4 5 8
5 6 7	3 6 7	2 5 10	2 6 8	1 8 9	1 5 7	1 6 10
8 9 10	4 9 10	4 7 8	3 5 9	4 5 6	3 8 10	2 7 9

Here the leave is a 6-cycle $(5, 9, 7, 8, 6, 10)$. □

Remark. Note that with this $\tilde{R}$MP$(3, 2; 10, 7)$, the proof of the existence of an $\tilde{R}$MP$(3, 2; n, n-3)$ can be simplified a lot, and the resulted $\tilde{R}$MP$(3, 2; n, n-3)$ have a 6-cycle as its leave. But in order to deal with the two designs $\tilde{R}$MP and $\tilde{R}$MC together, we do not use this small design in the recursive constructions.

Lemma 4.2. *There exists an* $\tilde{R}$MC$(3, 1; 67, 33)$.

Proof. Take the point set $V = (Z_{33} \times \{1, 2\}) \cup \{\infty\}$. Instead of listing all the blocks and the parallel classes of the desired designs, we only list the blocks of the initial parallel class. The 33 parallel classes will be generated mod 33 from it. The blocks of the initial parallel class are listed below. We write (a, i) as a_i for brevity.

$1_1 2_1 1_2 3_2$	$15_2 16_2 21_2$	$24_1 32_1 14_2$	$9_1 22_1 31_1$	$6_2 10_2 31_2$	$5_2 24_2 0_1$
$6_1 12_1 23_2$	$4_1 20_1 30_1$	$21_1 26_1 9_2$	$23_1 25_1 18_2$	$10_1 14_1 28_1$	$17_2 26_2 19_1$
$2_2 13_2 27_1$	$3_1 15_1 12_2$	$4_2 19_2 20_2$	$5_1 8_1 11_2$	$7_1 11_1 29_2$	$7_2 0_2 13_1$
$8_2 28_2 16_1$	$22_2 32_2 18_1$	$27_2 30_2 17_1$	$29_1 25_2 \infty$		

Here the excess is $\{i_1, (i+4)_1\} \cup \{i_2, (i+1)_2\}$, $i \in Z_{33}$. □

Therefore, there are five small designs, the existence of which remain undecided for $\tilde{R}$MP$(3, \lambda; n, m)$ and $\tilde{R}$MC$(3, \lambda; n, m)$, where $\lambda \in \{1, 2\}$.

	(3, 1)	(3,2)
$\tilde{R}$MP	19	13
$\tilde{R}$MC	16	10,13

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